COURSE ON ACOUSTICS

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7. Solving the acoustic field equations numerically - method of elements

7.1. **Finite element method (FEM) in acoustics.**

7.1.1. *The equation.* The purpose is to derive the element method equations for the Helmoholtz equation. Let $p(\mathbf{r})$ be the acoustic pressure in volume Ω with sound source $f(\mathbf{r})$.

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = f(\mathbf{r}). \tag{1}$$

On the boundary $\partial\Omega$ we use the *admittance* $\zeta(\omega)$ to set the boundary condition.

$$\frac{\partial p}{\partial \mathbf{n}} = \zeta(\omega) p = \frac{-i\rho\omega}{Z(\omega)} p \tag{2}$$

7.2. **Galerkin method.** The Galerkin method [44] starts by setting a finite function base, which approximates the sought after solution. The approximate solution is given in terms of a linear combination of these bases functions and solving the system falls back to inverting a finite matrix equation.

The method is based on a *weak form* of the integral equation in which the equation is first multiplied by a test function $\phi_i(\mathbf{r})$, and then integrated over the volume.

$$\int_\Omega \left[\nabla^2 p(\mathbf{r} + k^2 p(\mathbf{r}\right] \phi_i(\mathbf{r}) d\Omega = \int_\Omega f(\mathbf{r})\phi_i(\mathbf{r}) d\Omega \tag{3}$$

If the set of test functions ϕ_i is complete function base, the solution to the equation above will be a solution of the wave equation as well. As, everything is given under the integral sing, one could alter solution as any finite number of points without affecting the values of the integral. Also this would allow the solution to have kinks and even to be discontinuous. This is the reason to call Equation (refweakform)a weak form of the Helmholtz equation.

For practical reasons, let us choose the test functions among properly smooth functions, that span the full function space. Let us choose, although this is not a necessity, utilize the same function space for the trial of the pressure function itself.

By integrating in parts the weak form turns to

$$\oint_{\partial\Omega} \phi_i(\mathbf{r}) \frac{\partial p(\mathbf{r})}{\partial \mathbf{n}} d\Gamma + \int_\Omega \left[-\nabla p(\mathbf{r}) \nabla \phi_i(\mathbf{r}) + k^2 p(\mathbf{r}) \phi_i(\mathbf{r})\right] d\Omega = \int_\Omega f(\mathbf{r}) \phi_i(\mathbf{r}) d\Omega. \tag{4}$$

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On the boundary integral part we can use the (2) to replace the normal derivative of the pressure.

$$\oint_{\partial\Omega} \phi_i(\mathbf{r})\zeta p(\mathbf{r}) d\Gamma + \int_{\Omega} \left[-\nabla p(\mathbf{r})\nabla\phi_i(\mathbf{r}) + k^2 p(\mathbf{r})\phi_i(\mathbf{r})\right] d\Omega = \int_{\Omega} f(\mathbf{r})\phi_i(\mathbf{r}) d\Omega.s \tag{5}$$

Now, let us choose a discrete set of points, nodes, $\{bfr_i\} \in \Omega$, about evenly inside the domain. Assign an own test function ϕ_i to each of the nodes in such a way that

$$\phi_j(\mathbf{r}_i) = \delta_{ij} \tag{6}$$

So only one, unique, test function, is non-zero at each of the nodes. The condition is not restricting the test functions in other points. They may be local functions, as they have to have zeros on all the nodes except one, but this is not a fundamental requirement.

Let us try to find a solution for the problem as a linear combination of test functions.

$$p(\mathbf{r}) = \sum_i p_i \phi_i(\mathbf{r}), \tag{7}$$

Because of condition for the test functions (6), the linear coefficients of the linear combinations are actually the values of the pressure at mesh points:

$$p(\mathbf{r_i}) = p_i. \tag{8}$$

To get the values of the pressure in other points, we have to use the Equation (7). With the trial, our integral equation becomes

$$\sum_j p_j \left[\oint_{\partial\Omega} \phi_i(\mathbf{r})\zeta\phi_j(\mathbf{r}) d\Gamma + \int_{\Omega} \left[-\nabla\phi_j(\mathbf{r})\nabla\phi_i(\mathbf{r}) + k^2 p(\mathbf{r})\phi_i(\mathbf{r})\right] d\Omega\right] = \int_{\Omega} f(\mathbf{r})\phi_i(\mathbf{r}) d\Omega. \tag{9}$$

Suddenly, with known test functions ϕ_i, all the integrals in the equation are known. We have reduced the problem to a matrix equation.

$$\sum_j M_{ij} p_j = f_i, \tag{10}$$

where

$$M_{ij} = \oint_{\partial\Omega} \phi_i(\mathbf{r})\zeta\phi_j(\mathbf{r}) d\Gamma + \int_{\Omega} \left[-\nabla\phi_j(\mathbf{r})\nabla\phi_i(\mathbf{r}) + k^2 p(\mathbf{r})\phi_i(\mathbf{r})\right] d\Omega \tag{11}$$

and

$$f_i = \int_{\Omega} f(\mathbf{r})\phi_i(\mathbf{r}) d\Omega. \tag{12}$$

The next steps are then pure linear algebra, how to invert this matrix in a best possible way.

7.2.1. *Meshing.* Crucial skill in the practical work of solving the system is how to define the nodal points. When nodal points are selected, they are used to divide the space in sub volumes, elements (hence the name element models). Very often the elements have a defined geometry - cubic, tetraedric, pyramid, hexagonal,.... When the element is chosen, a test function, actually called *form function* is chosen- It may linear, quadratic, cubic,..... Complicated, high order form functions require multiple values for their full specification. Then, more nodal points, are required in an element. The higher functions are more accurate, but they lead to bigger matrices. Weather to just to increase the number of nodal points or use a more complicated form function is a matter of case by case optimization.

Usually about 10 elements are needed for a wavelength, minimum 7. This means that with a $N = 10000$ element mesh one can cover a systems of size s

$$s \approx \sqrt[3]{N}\frac{\lambda}{10} = \frac{\sqrt[3]{N}}{7}\frac{c}{f} \approx \frac{1000m}{f} \tag{13}$$

Hence, we can model a system that has a diameter one meter up to 1000Hz. There is a method that only deals with boundaries of the acoustic domain. The boundary element method. (BEM). With a similar mesh, one van get a little higher frequencies

$$s \approx \sqrt{N}\frac{\lambda}{10} = \frac{\sqrt{N}}{7}\frac{c}{f} \approx \frac{1500m}{f}, \tag{14}$$

where only the boundaries are meshed. This is an advantage as the meshing of the model is much easier. The time spent by the user is reduced significantly. The downsize is that the BEM matrix is not sparse like the FEM matrix, and iterative matrix solvers are slower for BEM. However, I guess, it is easier to let the computer run, even if a little longer.

Generally, the solving time increases faster than the number on elements. This is because the matrix becomes more and more singular as elements are added. Some preconditioning methods are essential in practice.

7.3. **Boundary element methods.** Bondary element methods are based on the integral formulation of pressure we already derived in (3-21). In the 1960's multiple numerical codes based on integral representations were developed for flow and structure prediction. The common name for these methods was coined as late as in 1977. For acoustics, the problem can be formulated in various ways. The pressure is calculated either outside or inside the structure. In direct BEM (collocation) is the simpler version conceptually, but the values of pressure can be derived only on the outside or inside of the geometry defined by the boundary. In indirect BEM, there may be holes in the surface and pressure can be calculated at the same time on both sides of the structure. For simplicity we consider here only the collocation version.

7.3.1. *Theory.* Let us look at the integral equation (3-21), but without the use of the δ-function property

$$\int_\Omega p(\mathbf{r})\delta(\mathbf{r}-\mathbf{r_0})d\Omega = \int_\Omega G(\mathbf{r}-\mathbf{r_0})f(\mathbf{r})d\Omega + \int_{\partial\Omega} [G(\mathbf{r}-\mathbf{r_0})\nabla p(\mathbf{r}) - p(\mathbf{r})\nabla G(\mathbf{r}-\mathbf{r_0})]\cdot\mathbf{n}dS, \tag{15}$$

where $G(\mathbf{r}-\mathbf{r_0})$ is the Green's function in free space.

$$G(\mathbf{r}-\mathbf{r_0}) = \frac{1}{4\pi}\frac{\exp[ik|\mathbf{r}-\mathbf{r_0}|]}{|\mathbf{r}-\mathbf{r_0}|}, \tag{16}$$

As

$$p(\mathbf{r}_0) = \int_\Omega p(\mathbf{r})\delta(\mathbf{r}-\mathbf{r_0})d\Omega \tag{17}$$

inside the domain Ω, we know pressure anywhere inside the domain, provided it is known on the boundaries. To make a self-consistent equation out of this, we need to know be able to apply this on the boundary $\partial\Omega$of the volume. This requires, integrating over the δ-function on the surface. The the value of the integral depends on how big a portion of the total solid angle is visible from the boundary point to the acoustic media. For points inside the domain the solid angle is 4π, and for a flat surface it would be 2π. For any point P on a surface $\partial\Omega$ it can be calculated from the integral

$$4\pi C(P) \equiv \int_{\partial\Omega} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS, \tag{18}$$

Therefore, we have

$$C(P)p(\mathbf{r}_0) = \int_{\Omega} G(\mathbf{r}-\mathbf{r_0})f(\mathbf{r})d\Omega + \int_{\partial\Omega} [G(\mathbf{r}-\mathbf{r_0})\nabla p(\mathbf{r}) - p(\mathbf{r})\nabla G(\mathbf{r}-\mathbf{r_0})]\cdot \mathbf{n} dS, \tag{19}$$

Now, the pressure on the left side is also on the boundary. This is a self-consistency equation for the pressure on the boundary.

In collocation method each element Γ_i on the boundary is assumed to have a constant value of pressure p_k and velocity

$$i\rho_0\omega v_i = \frac{\partial p}{\partial n} \tag{20}$$

The velocity and pressure are related on the surface by the impedance of the surface. In a general form this is in each element i

$$\alpha_i p_i + \beta_i v_i = \gamma_i. \tag{21}$$

Different choices of α_i, β_i, and γ_i cover all local boundary coniditions: Neumann, Dirichlet or mixed, impedance boundary condition.

As the velocities and pressures are constants in each elements, the integral equation has the form

$$C(\Gamma_k)p_k = \int_{\Omega} G(\mathbf{r}-\mathbf{r_k})f(\mathbf{r})d\Omega + \sum_i \left[i\rho_0\omega v_i \int_{\Gamma_i} G(\mathbf{r}-\mathbf{r_k})d\Gamma_i - p_i \int_{\Gamma_i} \nabla G(\mathbf{r}-\mathbf{r_k})\cdot \mathbf{n} d\Gamma_i \right], \tag{22}$$

Now, the equations (21) and (22) make a linear matrix equation for p_i and v_i that can be solved by matrix inversion.

7.4. **Modern solutions methods.** In finite element methods it is important to choose the basis functions in a proper way, so that the waves inside each element can be described adequately inside each element. At the extreme, one may think that one could add, inside each element, a vast amount of plane waves, in all directions. The *Ultra Weak Variational Method* is based on this idea. It has been applied to acoustics here in Finland in the University of Kuopio [45]. This method was used in Nokia Research Center to simulate, for the first time, the head related transfer functions (HRTF) for the full audio range up to 25kHz [2]. Because each elements contain already waves, one can use a sparser grid. However, the number of directions of plane waves may be up to 130 in an element. The ultra weak method is a finite element method.

Another modern way is to improve the BEM: *Multilevel Fast Multipole Algorithm* (MLFMA). This method has been developed at Aalto University To be able to understand the idea one has to peek to the soul of BEM. As the sound is reflected from the surface to the other surfaces, it may travel through the medium very far from the element that reflected it. There is no locality. However, in multipole methods, one uses the distance to advantage. For far away sources, the effect of far away close-by elements is very similar and can be summed together before considering their long journey back to home. At short distances this will not work, and we have to look at each boundary element individually. Advanced methods can handle meshes that contain themselves different sized of elements,[46]. The method is not very far from the idea of Fast Fourier Transformation. Indeed, the summing of terms at far changes the matrix multiplication time from order N^2 to order $N \log N$. This method is in active use in day to day audio design at Nokia.

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